Waves and Optics midterm exam - with answers
16-12-2024 11:00-13:00

6 questions, 30 points

This 2-hour midterm exam tests your understanding of the main concepts of the Waves and Optics course. The last page of the exam is a formula sheet. Write your name and student number on each page of your answer sheet!

1. **Electromagnetic waves**

   (a) (2 points) Explain all terms on the right-hand side of the electromagnetic wave equation in a medium. *The first term on the right-hand side of the equation describes currents of free charges, which are important for determining the reflection of light from a metallic surface or for determining the propagation of light in a plasma. The second term on the right-hand side describes dipole oscillations, which behave similar to currents. These dipole oscillations play a prominent role when light propagates in nonconducting materials. The final term on the right-hand side of (1.40) is important in anisotropic media such as crystals. In this case, the polarization P responds to the electric field along a direction not necessarily parallel to E, due to the influence of the crystal lattice.*

   (b) (1 point) Give an example of a material where at least two of the terms on the right-hand side are non-zero, and explain why. *A transparent crystal typically has both a non-zero anisotropy and oscillating dipoles - so the second and third term are non-zero.*

2. **Interferometers and coherence**.

   (a) (2 points) Explain what is needed to obtain a good contrast using a Michelson-Morley interferometer with a white light source. *Note, that the question is specific to white light. You need the path length in the two arms to be (close to) identical, since the coherence length of white light is very short (full points). If the two arms are exactly the same length, this does not matter and good contrast is visible. 50/50 beam splitter is assumed in a interferometer, and in case we have white light and not monochromatic. Coherence 0.5 (implies path length dependence indirectly).*

   (b) (2 points) Explain what is needed to obtain good contrast using a double-slit interferometer with a white light source. *Note, that the question is specific to white light. A double-slit interferometer probes the transverse spatial coherence of the wavefront. You get good contrast in the interferometer if the wavefront is sufficiently close to a plane wave; this is true for a point source that is sufficiently far away, even if it is a white light source. Plane wave ,or temporally AND spatially coherent 2pt. Correct spacing and width of slits is a requirement for interference, but then must specify the conditions 0.5pt (assumed in setup). Just "Good" for conditions is not accepted. Light source far away (without further physical reasoning) 0.5pt.*

(c) (3 points) What is the transmission spectrum when you shine a white light source onto a Fabry-Perot cavity? Make a sketch of the transmission spectrum, and explain using the transmission equation on the formula sheet how (if at all) this is different from the input spectrum. *The cavity will have high transmission for particular wavelengths - if* $\sin^2(\frac{4\pi nd}{2\lambda})$ *is equal to zero. A non-zero* ϕ_r *will just give a shift to the whole spectrum. A resonance occurs each time that* $\frac{nd}{\lambda}$ *is a multiple of* $\frac{1}{2}$*. The output spectrum therefore looks like a comb. For* $n \approx 1$ *(air) and* $d = 10$ *cm, and the wavelength region around 500 nm, these maxima occur about 200 times per nm. Since (depending on the reflectivity of the mirrors) most of the light does not fulfill these criteria, the transmitted spectrum has a greatly reduced intensity.*

3. **Polarization, waveplates and polarizers**. The glasses that are used for 3D movies in the cinema consist of the combination of a quarter-wave plate and a linear polarizer.

(a) (2 points) Explain how an anisotropic material like a crystal can be used to create a quarter-wave plate. *A quarter-wave plate is made from a birefringent material, where light polarized along two principal axes experiences different refractive indices, causing a relative phase shift [1pt]. The two principal axes are called the fast and slow axis. By carefully choosing the plate's thickness, the phase shift difference between the components of light traveling along these axes becomes* $90°$ *[0.5pts]. In these conditions the waveplate converts linearly polarized light into circularly polarized light or vice versa [0.5pts].*

(b) (2 points) Show, using the polarization vector in combination with the appropriate Jones matrices, what happens when an incoming left-handed circularly polarized electromagnetic wave passes through a quarter-wave plate. Make sure to address the effect of the orientation of quarter-wave plate. *Multiply the quart-wave plate matrix with the polarization vector for left-handed circular polarization (given in the formula sheet) to the general expression. We can find that for* $\theta = 45°$*, this reduces to horizontally polarized light. (0.5 for correct formula taken, 0.5 for correct math, 0.5 for note* $\theta = \frac{\pi}{4}$ *case, 0.5 for correct interpretation.)*

(c) (2 points) You are tasked with the construction of such glasses, and you need to make sure one glass transmits only right-handed polarized light, and the other glass transmits only left-handed polarized light. Show how you would combine the quarter-wave plate with the linear polarizer for each of the glasses to achieve this effect. *From the previous answer, we see that when the quarter-wave plate is followed by a horizontal polarizer, it only transmits left-handed circularly polarized light. If followed by a vertical polarizer, it only transmits right-hand circularly polarized light. 0.5 pt for order and 0.5 pt for orientation for each glass.*
However, we found that many of you struggled with this question because it was unclear. It was not clear wether the input was a combination of right-hand left-handed circularly polarized light that was to be filtered by the glasses (as in the 3D cinema, and as intended in the question) or that the glasses should be designed such to only produce circularly polarized light from whatever the input light was. Therefore, we have decided to give full points for this subquestion regardless of the answer.

4. **Complex index of refraction, Lorentz model, crystals**

(a) (2 points) Explain all terms in the equation of motion of the electrons in the Lorentz model, and sketch a graph of the real and complex parts of the wavelength-dependent index of refraction as a function of the frequency of the electromagnetic wave. *The electric field pulls on the electron with force* $q_e\mathbf{E} = m_e\ddot{\mathbf{r}}$ *(0.5). A drag force (or friction)* $\gamma\dot{\mathbf{r}}_e$ *opposes the electron motion and accounts for absorption of energy (0.5). The term* $\omega_0^2\mathbf{r}$ *corresponds to the resonance as a result of the restoring force on the electron towards the nucleus (0.5). Graph drawn correctly, see below (0.5).*

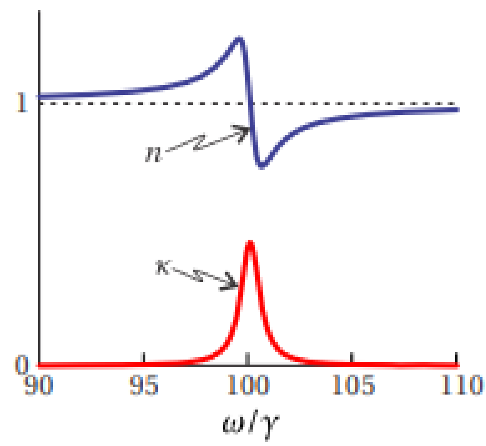
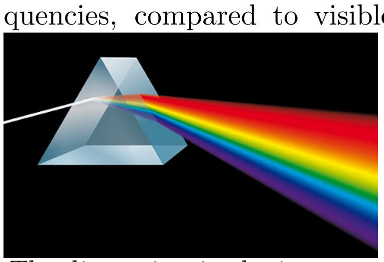


Figure 1: Real and imaginary parts of refractive index

(b) (2 points) Is the main resonance frequency of glass at higher frequencies or lower frequencies, compared to visible light? To answer this question, use the image below.



The dispersion in the image can be seen to be increasing towards shorter wavelengths, which have higher frequencies (0.6). This is what you would see at the lower-frequency side of a resonance (0.7). So the resonance frequency in glass is at a high frequency compared to visible light (0.7, correct conclusion). (0.7 points if only final answer is given)

5. **Diffraction**

(a) (2 points) Explain the connection between Fourier transforms and diffraction patterns. Make sure to mention the approximations that allow you to make this connection.
Due to the diffraction of light, when sending a coherent light beam through a small aperture, the pattern it projects on a screen far away needs to be considered as a superposition of the "light beams" that move through each point on the aperture. Mathematically, this means that the projection on the screen is described by a two-dimensional integral over the aperture's surface, and the integrant propagated along the distance between the aperture and the screen. In the Fraunhofer limit, this expression becomes identical to a two-dimensional Fourier transform. Physically, this means that the decomposition of the frequency components of the aperture become visibly projected onto the screen.

(b) (2 points) For the images below, connect the masks (A-E) with their associated diffraction pattern (0-4).

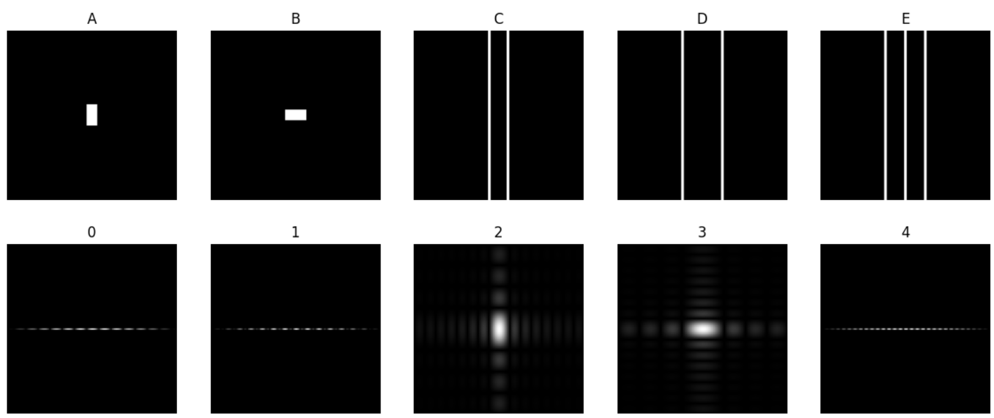
Answer: The correct combinations are A3, B2, C0, D4, E1.

For A3 and B2 it is important to remember that the more confined a pattern becomes in some direction (say, horizontal), the less confined it becomes in the Fourier transform along this direction, and vice versa.

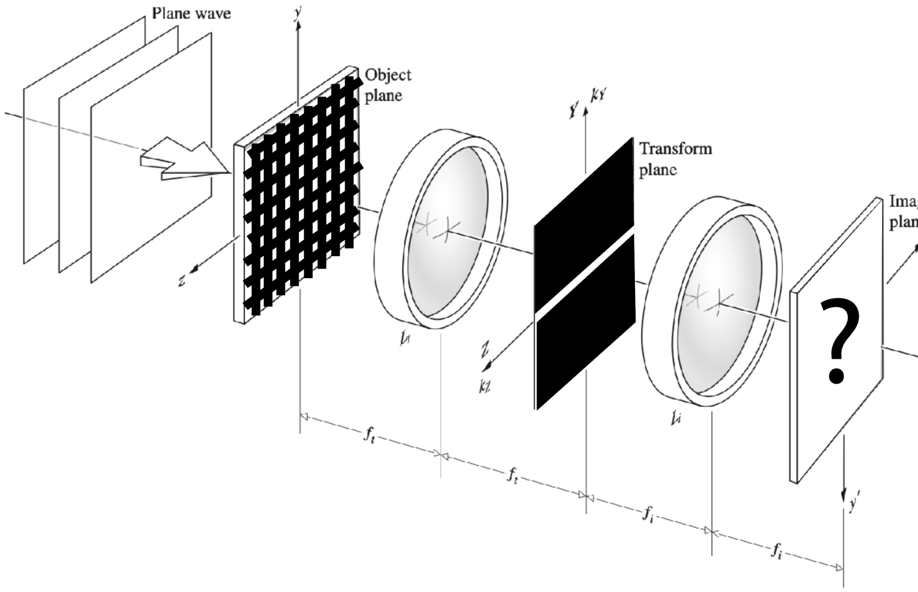
Similarly, for C0 and D4: the distance between the slits increases from C0 to D4, so the Fourier transform becomes more narrow.

From C0 to E1: adding more slits makes the pattern more defined (bright spots and dark regions, rather than a gradual change). Eventually, adding many slits creates a grating.

0.4 points for each correct answer.



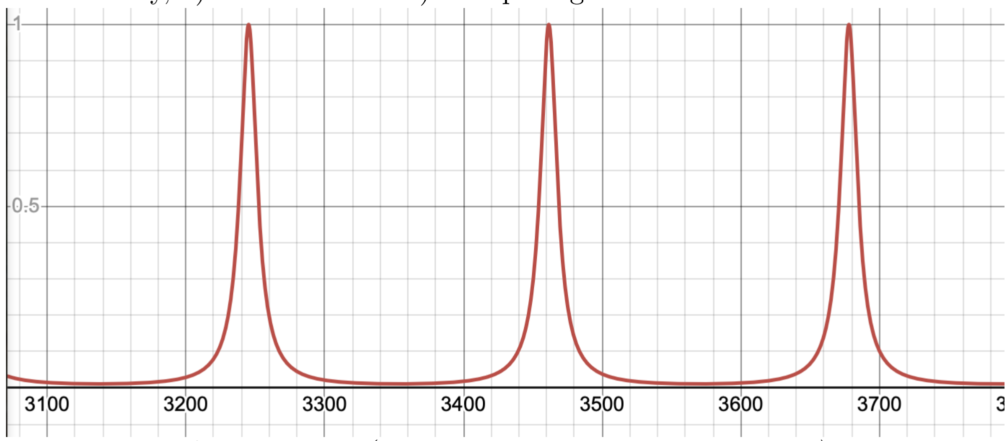
(c) (2 points) Imagine an electromagnetic plane wave, which encounters a transmission pattern as indicated in the object plane in the image below. The diffracted light is collected by a lens. Downstream from the lens, in the transform plane a horizontal slit is placed. A second lens collects the light transmitted through this slit, and images that onto a screen at the image plane. What does the image on the screen look like? Make a sketch, and *explain how and why* this pattern (if at all) is different from the original pattern.



The object can be seen as a combination of a horizontal and a vertical grating. The horizontal slit in the transform plane transmits all diffraction peaks of the vertical grating. Of the horizontal grating, only the central zero-th order diffraction peak is transmitted, all others are filtered out; therefore, the horizontal grating is no longer visible in the image plane, and we only see vertical stripes there. It is accepted if one assumes the grating was small enough to cause diffraction such that a diffraction pattern is imprinted on the vertical stripes; however, this was not required for full points.

1.5 points for missing only one component for the correct answer (i.e., the image either does not have repeating vertical stripes (just one vertical stripe), has repeating vertical stripes with decreasing intensity from the center, or repeating horizontal stripes. 1 point for a single horizontal stripe or horizontal stripes with decreasing intensity. 0.5 points for missing two components of the correct answer besides the previously mentioned ones. 0 points for missing three or more components of the answer (like including both horizontal and vertical stripes). (Written answers are treated as correct only if the explanation is very clear. Otherwise, point deductions are made for not including a sketch.)

6. **Graphs of interferometer transmission**. The graph below plots the total transmitted intensity T_{total} of a Fabry-Perot cavity, as a function of the phase ϕ. Explain *how and why* 1) the intensity, 2) the width and 3) the spacing of the transmission maxima change when...



Each aspect of the answer (i.e., intensity, width, and spacing) contribute 0.4, 0.3, and 0.3 points. Irrespective of which aspect is wrong, the first point deduction beings with 0.4 and continues until 0 points. (The answers are still accepted if the numbers are not specified.)
We found an issue in the formulation of subquestions (a) and (d), which were not clear because the transmitted intensity is actually plotted as a function of the spacing. This is possible because the phase itself is a function of mirror spacing, index of refraction and the wavelength of the light. Therefore, we have decided to give full points to 6a) and 6d) regardless of the answer.

(a) (1 point) ...the wavelength of the light is increased by 10% *1) intensity: no change, 2) width: no change, 3) spacing: increased by 10% because the longer wavelength requires a larger distance to accumulate the same phase difference.*

(b) (1 point) ...the intensity of the light is increased by 10%
1) intensity: no change (since normalized), 2) width: no change, 3) spacing: no change

(c) (1 point) ...the reflectivity of the mirror is increased by 10%
1) intensity: no change, 2) width: decreases, since the Finesse increases, which gives the amplitude of the modulating term in the transmission equation, 3) spacing: not changed

(d) (1 point) ...instead of air between the mirrors we have a gas with a 10% larger index of refraction *1) intensity: no change, 2) width: no change, 3) spacing: decreased by 10%, since now the phase ϕ accumulates faster for a given distance and wavelength.*

Formula sheet

Electromagnetic wave equation in a medium

$\nabla^2 \mathbf{E} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial \mathbf{J}_{\text{free}}}{\partial t} + \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2} - \frac{1}{\epsilon_0} \nabla(\nabla \cdot \mathbf{P})$

Transmission of a Fabry-Perot cavity:

$T_{\text{total}} = \frac{T_{\max}}{1+F\sin^2\left(\frac{\phi}{2}\right)}$ with $T_{\max} = \frac{T^2}{(1-R)^2}$, $F = \frac{4R}{(1-R)^2}$, $\phi = \frac{4\pi n d}{\lambda} \cos(\theta_i) + 2\phi_r$

Jones matrices:

linear polarizer: $\begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$,

half-wave plate: $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$,

quarter-wave plate: $\begin{bmatrix} \cos^2\theta + i\sin^2\theta & (1-i)\sin\theta\cos\theta \\ (1-i)\sin\theta\cos\theta & \sin^2\theta + i\cos^2\theta \end{bmatrix}$

Right circular polarizer: $\frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$,

Left circular polarizer: $\frac{1}{2}\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$.

Polarisation vector

$\begin{bmatrix} A \\ Be^{i\delta} \end{bmatrix}$, where $\mathbf{E}(z,t) = E_{\text{eff}}\left(A\hat{\mathbf{x}} + Be^{i\delta}\hat{\mathbf{y}}\right) e^{i(kz-\omega t)}$.

Left handed circularly polarized light: $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$, right handed circularly polarized light: $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$

Lorentz model

$\ddot{\mathbf{r}}_e + \gamma\dot{\mathbf{r}}_e + \omega_0^2 \mathbf{r}_e = \frac{q_e}{m_e}\mathbf{E}$, which leads to $\mathbf{P} = \epsilon_0 \left(\frac{\omega_p^2}{\omega_0^2 - i\omega\gamma - \omega^2}\right) \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$

and $(n + i\kappa)^2 = 1 + \chi(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - i\omega\gamma - \omega^2}$

Diffraction

Huygens-Fresnel : $E(x,y,z) = -\frac{i}{\lambda}\iint_{\text{aperture}} E(x',y',0)\frac{e^{ikR}}{R}dx'dy'$ with $R = \sqrt{(x-x')^2 + (y-y')^2 + z^2}$

Fraunhofer: $E(x,y,z) \approx -\frac{ie^{ikz}e^{i\frac{k}{2z}(x^2+y^2)}}{\lambda z}\iint_{\text{aperture}} E(x',y',0)e^{-i\frac{k}{z}(xx'+yy')}dx'dy'$